INVERTED EXPONENTIAL DISTRIBUTION AS A LIFE DISTRIBUTION MODEL FROM A BAYESIAN VIEWPOINT

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ABSTRACT

The Inverted Exponential Distribution is studied as a prospective life distribution. In this paper, we derive Bayes' estimators for the parameter θ of inverted exponential distribution. These estimators are obtained on the basis of squared error and LINEX loss functions. Comparisons in terms of risks with the estimate of θ under squared error loss and LINEX loss functions have been made. Finally, numerical study is given to illustrate the results.

Keywords: Bayes' estimator, LINEX loss function, Reliability function, Risk function, Squared error loss function.

1 INTRODUCTION

In reliability studies commonly used models in life testing include the gamma, lognormal and inverse Gaussian distributions. These models are usually chosen on the basis of what is understood about the failure mechanisms. If the failures are mainly due to aging or the wearing out process, then it is reasonable in many applications to choose one of the above mentioned distributions (see Chhikara & Folks, 1977; Sinha & Kale, 1980; Von Alven (ed.), 1964; Sherif & Smith, 1980). In this paper, we consider the inverted exponential distribution as life distribution (see Lin., Duran, & Lewis, 1989). The probability density function (pdf) of the inverted exponential distribution with parameter θ is

$$f(x;\theta) = \frac{1}{\theta x^2}\exp(-\frac{1}{x\theta}) \quad ; x > 0 \ , \ \theta > 0 \tag{1}$$

$= 0$, otherwise,

which has no finite moments.

The reliability function, i.e., the probability of no failure before time 't' is

$$R(t) = 1 - F(t) = 1 - e^{-\frac{1}{t\theta}}$$

where F(t) is the distribution function of X.

The failure rate of an inverted exponential distribution with parameter θ is

$$r(t) = \frac{f(t)}{R(t)} \ , \ t > 0$$

$$= \frac{\frac{1}{\theta t^2} e^{-\frac{1}{t\theta}}}{1 - e^{-\frac{1}{t\theta}}} \tag{2}$$

In the estimation of reliability function, use of symmetric loss function may be inappropriate as has been recognized by Canfield (1970) and Varian (1975). Zellner (1986) proposed an asymmetric loss function known as the Linex loss function which has been found to be appropriate in the situation where overestimation is more serious than underestimation or vice-versa.

Suppose $\Delta = \frac{\hat{\theta}}{\theta} - 1$, where $\hat{\theta}$ is an estimate of θ. Consider the following convex loss function.

$$L(\Delta) = e^{a\Delta} - a\Delta - 1;\ a \neq 0 \tag{3}$$

The sign and magnitude of 'a' represent, respectively, the direction and degree of asymmetry. A positive value of 'a' is used when overestimation is more costly than underestimation; while a negative value of 'a' is used in the reverse situation. For 'a' close to zero, this loss function is approximately squared error loss and therefore almost symmetric. Several authors (Basu & Ebrahimi, 1991; Rojo, 1987; Soliman, 2000; Zellner, 1986) have used this loss function in various estimation and prediction problems.

If we define $\Delta_1 = \hat{\theta} - \theta$, then $L(\Delta_1)$ is equivalent to the loss function used by Varian (1975) and Zellner (1986).

Here, we consider the non-informative prior:

$$g(\theta) \propto \frac{1}{\theta}, \quad \theta>0 \tag{4}$$

The plan of the article is as follows: In section 2, we obtain Bayes estimator of θ. The estimates are based on the squared error loss function and Linex loss function $L(\Delta_1)$ where $\Delta_1 = \hat{\theta} - \theta$. By using g(θ) as the prior distribution, the risk of estimates have been obtained. Comparison in terms of risk with the estimates of θ under squared error loss and Linex loss functions have been made. Also, we give a numerical example to compare our results.

2 BAYES' ESTIMATE OF θ

In this section we are concerned with the estimation of the unknown parameter θ of the inverted exponential distribution based on a complete random sample of size n. The likelihood function (LF) is given by:

$$L(x|\theta) = \frac{1}{\theta^n} \prod_{i=1}^{n} \frac{1}{x_i^2} e^{-\frac{S}{\theta}} \tag{5}$$

where, $S = \sum_{i=1}^{n} \frac{1}{x_i}$

The natural logarithm of the LF (5) is

$$\ell = \ell n L(x \mid \theta) = \text{Const} + \ell n \prod_{i=1}^{n} \frac{1}{x_i^2} - n\,\ell n\theta - \frac{s}{\theta}$$

Assuming that the parameter θ is unknown, the MLE (Maximum Likelihood Estimator) of the parameter θ can be shown to be

$$\hat{\theta}_{ML} = \frac{S}{n} \tag{6}$$

because $\frac{1}{x_i}$; i = 1, 2, . . . , n, are independent and identically distributed exponential random variables with parameter θ. It is also known that a sum of independent exponentially distributed random variables gives a Gamma distributed variable where the probability density function of S is

$$h(S) = \frac{1}{\theta^n} S^{n-1} e^{-\frac{S}{\theta}} ;\ S > 0 \tag{7}$$

2.1 Bayes' Estimator of θ based on squared error loss function

Combining the prior distribution g(θ),with the likelihood function L(x|θ) using Bayes' theorem, the posterior density of θ is

$$\pi(\theta \mid x) = \frac{S^n e^{-\frac{S}{\theta}}}{\theta^{n+1} \Gamma n}, \quad \theta > 0 \qquad (8)$$

By using (8) under squared error loss (L($\hat{\theta},\theta) = (\hat{\theta}-\theta)^2$), the Bayes' estimator of θ denoted by $\hat{\theta}_{SB}$ is the posterior mean $E_\pi(\theta)$:

$$\hat{\theta}_{SB} = \left(\frac{S}{n-1}\right) \qquad (9)$$

2.2 Bayes' Estimator of θ based on LINEX Loss Function

Under the Linex loss function (1.3), the posterior expectation of the loss function L(Δ_1) with respect to $\pi(\theta \mid x)$ in (8) is

$$E[L(\Delta_1)] = \int_0^\infty \{ e^{a[(\frac{\hat{\theta}}{\theta})-1]} - a[(\frac{\hat{\theta}}{\theta}) - 1] - 1\} \, \pi(\theta \mid x) d\theta \qquad (10)$$

$$= e^{-a} E[\exp\{a(\frac{\hat{\theta}}{\theta})\}] - a E[(\frac{\hat{\theta}}{\theta}) - 1] - 1 \qquad (11)$$

The value of $\hat{\theta}$ that minimizes the posterior expectation of the loss function L(Δ_1) denoted by $\hat{\theta}_{LB}$ is obtained by solving equation

$$\frac{\partial E[L(\Delta_1)]}{\partial \hat{\theta}} = E_\pi[e^{-a} \frac{a}{\theta} \exp(a(\frac{\hat{\theta}}{\theta}))] - aE_\pi(\frac{1}{\theta}) = 0 \qquad (12)$$

that is, $\hat{\theta}_{LB}$ is the solution of the equation

$$E_\pi[\frac{1}{\theta}\exp(a(\frac{\hat{\theta}_{LB}}{\theta}))] = e^a E_\pi(\frac{1}{\theta}) \qquad (13)$$

provided that all expectation exists and are finite. Using (8) and (13), we get the optimal estimate of θ relative to L(Δ_1):

$$\hat{\theta}_{LB} = \frac{S}{a}[1 - e^{-\frac{a}{n+1}}] \qquad (14)$$

2.3 The Risk Efficiency of $\hat{\theta}_{LB}$ with respect to $\hat{\theta}_{SB}$ under LINEX Loss $L(\Delta_1)$

The risk function of estimators $\hat{\theta}_{LB}$ and $\hat{\theta}_{SB}$ relative to $L(\Delta_1)$ are of interest. These risk functions are denoted by $R_L(\hat{\theta}_{LB})$ and $R_L(\hat{\theta}_{SB})$, where subscript L denotes risk relative to $L(\Delta_1)$ and are given by using h(S) in (7) as follows:

$$R_L(\hat{\theta}_{LB}) = E_X(L(\Delta_1)] = \int_0^\infty \{e^{a[(\frac{\hat{\theta}_{LB}}{\theta})-1]} - a[(\frac{\hat{\theta}_{LB}}{\theta}) - 1] - 1\} \text{ h(s)ds}$$

$$= e^{-\frac{a}{n+1}} - n(1 - e^{-\frac{a}{n+1}}) + a - 1 \tag{15}$$

In the same manner, we get

$$R_L(\hat{\theta}_{SB}) = E_X(L(\Delta_1)] = \int_0^\infty \{e^{a[(\frac{\hat{\theta}_{SB}}{\theta}) - 1]} - a[(\frac{\hat{\theta}_{SB}}{\theta})^2 - 1] - 1\} \text{ h(s) ds}$$

$$= e^{-a}(1 - \frac{a}{n-1})^{-n} - \frac{a}{n-1} - 1 \tag{16}$$

The risk efficiency of $\hat{\theta}_{LB}$ with respect to $\hat{\theta}_{SB}$ under LINEX Loss $L(\Delta_1)$ may be defined as follows:

$$RE_L(\hat{\theta}_{LB}, \hat{\theta}_{SB}) = \frac{R_L(\hat{\theta}_{SB})}{R_L(\hat{\theta}_{LB})} \tag{17}$$

2. 4 The Risk Efficiency of estimators $\hat{\theta}_{LB}$ with respect to $\hat{\theta}_{SB}$ under squared error loss

The risk functions of the estimators $\hat{\theta}_{LB}$ and $\hat{\theta}_{SB}$ under squared error loss are denoted by $R_S(\hat{\theta}_{LB})$ and $R_S(\hat{\theta}_{SB})$ and are given by :

$$R_S(\hat{\theta}_{LB}) = \int_0^\infty (\hat{\theta}_{LB} - \theta)^2 h(S)dS$$

Thus,

$$R_S(\hat{\theta}_{LB}) = \theta^2[\frac{n(n+1)}{a^2}(1 - e^{-\frac{a}{n+1}})^2 - \frac{2n}{a}(1 - e^{-\frac{a}{n+1}}) + 1] \tag{18}$$

$$R_S(\hat{\theta}_{SB}) = \int_0^\infty (\hat{\theta}_{SB} - \theta)^2 h(S)dS$$

Thus,

$$R_S(\hat{\theta}_{SB}) = \theta^2[\frac{n(n+1)}{(n-1)^2} - \frac{2n}{n-1} + 1] \tag{19}$$

The efficiency of $\hat{\theta}_{LB}$ with respect to $\hat{\theta}_{SB}$ under squared error loss is defined as:

$$RE_S(\hat{\theta}_{LB}, \hat{\theta}_{SB}) = \frac{R_S(\hat{\theta}_{SB})}{R_S(\hat{\theta}_{LB})} \quad (20)$$

3 NUMERICAL EXAMPLE

To compare the proposed estimator $\hat{\theta}_{LB}$ with the estimator $\hat{\theta}_{SB}$, the risk functions are computed so as to see whether $\hat{\theta}_{LB}$ out performs $\hat{\theta}_{SB}$ under LINEX loss L(Δ_1) and how $\hat{\theta}_{LB}$ performs as compared to $\hat{\theta}_{SB}$ when true loss is squared error. A comparison of this type may be needed to check whether an estimator is inadmissible under some loss function. If so, the estimator would not be used for the losses specified by that loss function. For this purpose the risks of the estimators and risk efficiency have been computed. Since one sample does not tell us much, so we generated N=500 samples of sizes n=10, 20, 30 from (1) with θ=1. The results are presented in Table 1 to Table 6.

Table 1. The Estimators $\hat{\theta}_{LB}$, $\hat{\theta}_{SB}$, the risk efficiencies $RE_L(\hat{\theta}_{LB}, \hat{\theta}_{SB})$ and $RE_S(\hat{\theta}_{LB}, \hat{\theta}_{SB})$ under the prior g(θ) for the values of a=2

n	$\hat{\theta}_{SB}$	$\hat{\theta}_{LB}$	$R_L(\hat{\theta}_{LB})$	$R_L(\hat{\theta}_{SB})$	$R_S(\hat{\theta}_{LB})$	$R_S(\hat{\theta}_{SB})$	$RE_L(\hat{\theta}_{LB}, \hat{\theta}_{SB})$	$RE_S(\hat{\theta}_{LB}, \hat{\theta}_{SB})$
10	1.1042	.8261	.1713	.4483	.0973	.1358	2.6170	1.3928
20	1.0500	.9062	.0923	.1464	.0496	.0582	1.5861	1.1734
30	1.0110	.9159	.0631	.0856	.0332	.0369	1.3566	1.1114

Table 2. The Estimators $\hat{\theta}_{LB}$, $\hat{\theta}_{SB}$, the risk efficiencies $RE_L(\hat{\theta}_{LB}, \hat{\theta}_{SB})$ and $RE_S(\hat{\theta}_{LB}, \hat{\theta}_{SB})$ under the prior g(θ) for the values of a=1

n	$\hat{\theta}_{SB}$	$\hat{\theta}_{LB}$	$R_L(\hat{\theta}_{LB})$	$R_L(\hat{\theta}_{SB})$	$R_S(\hat{\theta}_{LB})$	$R_S(\hat{\theta}_{SB})$	$RE_L(\hat{\theta}_{LB}, \hat{\theta}_{SB})$	$RE_S(\hat{\theta}_{LB}, \hat{\theta}_{SB})$
10	1.1146	.8718	.0441	.0835	.0927	.1358	1.8934	1.4649
20	1.0434	.9219	.0234	.0321	.0481	.0582	1.3718	1.2099
30	1.0240	.9427	.0160	.0197	.0325	.0369	1.2319	1.1354

Table 3. The Estimators $\hat{\theta}_{LB}$, $\hat{\theta}_{SB}$, the risk efficiencies $RE_L(\hat{\theta}_{LB}, \hat{\theta}_{SB})$ and $RE_S(\hat{\theta}_{LB}, \hat{\theta}_{SB})$ under the prior g(θ) for the values of a= 0.5

n	$\hat{\theta}_{SB}$	$\hat{\theta}_{LB}$	$R_L(\hat{\theta}_{LB})$	$R_L(\hat{\theta}_{SB})$	$R_S(\hat{\theta}_{LB})$	$R_S(\hat{\theta}_{SB})$	$RE_L(\hat{\theta}_{LB}, \hat{\theta}_{SB})$	$RE_S(\hat{\theta}_{LB}, \hat{\theta}_{SB})$
10	1.0947	.8756	.0112	.0186	.0914	.1358	1.6607	1.4858
20	1.0288	.9198	.0059	.0076	.0477	.0582	1.2881	1.2201
30	1.0053	.9329	.0040	.0047	.0323	.0369	1.1750	1.1424

Table 4. The Estimators $\hat{\theta}_{LB}$, $\hat{\theta}_{SB}$, the risk efficiencies $RE_L(\hat{\theta}_{LB},\hat{\theta}_{SB})$ and $RE_S(\hat{\theta}_{LB},\hat{\theta}_{SB})$ under the prior g(θ) for the values of a= - 0.5

n	$\hat{\theta}_{SB}$	$\hat{\theta}_{LB}$	$R_L(\hat{\theta}_{LB})$	$R_L(\hat{\theta}_{SB})$	$R_S(\hat{\theta}_{LB})$	$R_S(\hat{\theta}_{SB})$	$RE_L(\hat{\theta}_{LB},\hat{\theta}_{SB})$	$RE_S(\hat{\theta}_{LB},\hat{\theta}_{SB})$
10	1.1039	.9241	.0115	.0157	.0914	.1358	1.3652	1.4858
20	1.0399	.9522	.0059	.0069	.0477	.0582	1.1695	1.2201
30	1.0213	.9632	.0041	.0045	.0323	.0369	1.0976	1.1424

Table 5. The Estimators $\hat{\theta}_{LB}$, $\hat{\theta}_{SB}$, the risk efficiencies $RE_L(\hat{\theta}_{LB},\hat{\theta}_{SB})$ and $RE_S(\hat{\theta}_{LB},\hat{\theta}_{SB})$ under the prior g(θ) for the values of a= -1

n	$\hat{\theta}_{SB}$	$\hat{\theta}_{LB}$	$R_L(\hat{\theta}_{LB})$	$R_L(\hat{\theta}_{SB})$	$R_S(\hat{\theta}_{LB})$	$R_S(\hat{\theta}_{SB})$	$RE_L(\hat{\theta}_{LB},\hat{\theta}_{SB})$	$RE_S(\hat{\theta}_{LB},\hat{\theta}_{SB})$
10	1.1404	.9767	.0469	.0589	.0929	.1358	1.2559	1.4618
20	1.0690	.9906	.0242	.0271	.0482	.0582	1.1198	1.2075
30	1.0518	.9999	.0163	.0176	.0325	.0369	1.0797	1.1354

Table 6. The Estimators $\hat{\theta}_{LB}$,$\hat{\theta}_{SB}$, the risk efficiencies $RE_L(\hat{\theta}_{LB},\hat{\theta}_{SB})$ and $RE_S(\hat{\theta}_{LB},\hat{\theta}_{SB})$ under the prior g(θ) for the values of a= -2

n	$\hat{\theta}_{SB}$	$\hat{\theta}_{LB}$	$R_L(\hat{\theta}_{LB})$	$R_L(\hat{\theta}_{SB})$	$R_S(\hat{\theta}_{LB})$	$R_S(\hat{\theta}_{SB})$	$RE_L(\hat{\theta}_{LB},\hat{\theta}_{SB})$	$RE_S(\hat{\theta}_{LB},\hat{\theta}_{SB})$
10	1.1241	1.009	.1934	.2155	.0994	.1358	1.1143	1.3662
20	1.0455	.9924	.0983	.1036	.0499	.0582	1.0539	1.1663
30	1.0220	.9875	.0659	.0682	.0333	.0368	1.0349	1.1051

4 CONCLUSION

It is evident from the above Table 1 to Table 6 that, the risk efficiency $RE_L(\hat{\theta}_{LB},\hat{\theta}_{SB})$ is greater than one for the sample sizes n = 10, 20, 30 and for all values of ‘a’ (a = ± 0.5, ±1, ±2), which indicates that the proposed estimators $\hat{\theta}_{LB}$ is preferable to $\hat{\theta}_{SB}$ i.e., asymmetric loss function is more appropriate than squared error loss function. Also it is observed that, the risk efficiency $RE_L(\hat{\sigma}_{LB},\hat{\theta}_{SB})$ is greater than the risk efficiency $RE_S(\hat{\theta}_{LB},\hat{\theta}_{SB})$ for all positive values of ‘a’ and for all sample sizes n =10, 20, 30. However, the risk efficiency $RE_L(\hat{\sigma}_{LB},\hat{\theta}_{SB})$ is less than the risk efficiency $RE_S(\hat{\theta}_{LB},\hat{\theta}_{SB})$ for all negative values of ‘a’ and for all sample sizes n = 10, 20, 30.

5 ACKNOWLEDGEMENT

The author wishes to thank Mr. A.A. Basumatary, Department of Mathematics, St. Anthony's College, Shillong, for his assistance with computations.

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